INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 4611

Octobre 2002

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4611--FR+ENG

Optimal control of fluid-structure interaction systems : the case of a rigid solid

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet OPALE

Rapport de recherche n° 4611 — Octobre 2002 — 33 pages

Abstract: This report deals with problems arising in the optimal control for fluid-structure interaction systems. Our model consists of a fluid described by the incompressible Navier-Stokes equations interacting with an elastically supported rigid solid. We characterize the structure of the gradient of a cost function to be minimized using a new adjoint problem. The structure of the adjoint system is related to the use of an ALE transverse field introduced in the context of the minimization of non-cylindrical eulerian functionals

Key-words: Fluid-structure interaction, Navier-Stokes equations, ALE formulation, sensitivity analysis, transverse field, shape optimization, optimal control

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Contrôle optimal de systèmes couplés fluide-structure : cas du solide rigide

Résumé : Ce rapport a pour objet l'étude d'un problème de contrôle optimal pour un système mécanique constitué d'un solide rigide élastiquement supporté, dont le comportement est régi par une equation différentielle ordinaire, au sein d'un fluide newtonien en écoulement incompressible, dont l'évolution est régie par les équations de Navier-Stokes. Nous établissons la structure du gradient d'une fonctionnelle de coût à minimiser grâce à l'introduction d'un nouveau problème adjoint. La structure de cet adjoint est reliée à l'utilisation d'un champ transverse ALE intervenant dans le contexte de la minimisation de fonctionnelles eulériennes non-cylindriques.

Mots-clés : Interaction fluide-structure, équations de Navier-Stokes, formulation ALE, analyse de sensibilité, champ transverse, optimisation de forme, contrôle optimal.

Contents

1 Introduction

This report deals with the analysis of an inverse boundary problem arising in the study of bridge deck aeroelastic stability. The aeroelastic system consists of an elastically supported rigid solid moving inside an incompressible fluid flow in 2-D.
As described in [21], the stability analysis is performed by solving an instability tracking problem, which eventually leads to the determination of critical wind speeds of minimal energy that may produce unstable structural motions. Our work is focused on the mathematical justification of optimality conditions associated to the minimization problem involved in the instability tracking method.
The aeroelastic system we are dealing with, is described by a non-cylindrical system of partial differential equations, where the evolution of the moving boundary is unknown.
In case, where the evolution of boundaries is prescribed, inverse problems have been first considered by Da Prato *et al* [7] for a general parabolic equations written in non-cylindrical domains. The works of Wang [26] and Acquistapace *et al* [1] are closely related to the first one. The basic principle is to define a map sending the non-cylindrical domain into a cylindrical one. This process leads to the mathematical analysis of non-autonomous PDE's systems.
Recently a new methodology to obtain eulerian derivative for non-cylindrical functionals was introduced in [14]. This methodology was applied in [13],[12] to perform dynamical shape control in non-cylindrical Navier-Stokes equations where the evolution of the domain is the control variable.
For the aeroelastic coupled problem, there is a lack of results for the control case. The present report is an attempt to fill this gap. It is organized in four parts :

- In section 2, we introduce the notations and the mechanical system we shall deal with and we state the main result of this report, namely the structure of a cost functional gradient with respect to inflow boundary conditions pertubations. The proof is given in the remaining part of the paper.
- In section 3, we recall early results concerning the well-posedness of the coupled fluid-structure system.
- In section 4, we introduce the minimization problem and its associated Lagrangian functional.
- In section 5, we perform derivation of the Lagrangian with respect to state variables. This allows us to obtain the structure of the adjoint variables involved in the cost functional gradient.

2 Mathematical settings

We consider a two dimensionnal flexible structure in rigid motion. For the sake of simplicity, we only consider one degree of freedom for the structural motion : the vertical displacement

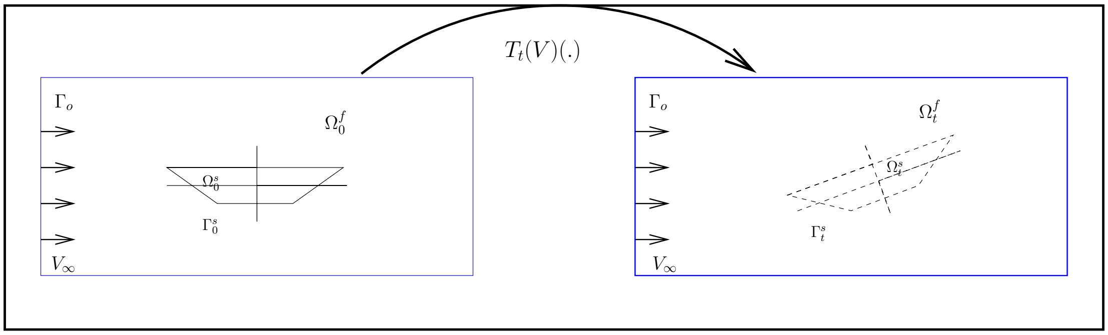

Figure 1: Arbitrary Euler-Lagrange map

$d(t)e_2$ where e_2 is the element of cartesian basis (e_1, e_2, e_3) in $\mathbb{R}^3$.
The structure is surrounded by a viscous fluid in the plane (e_1, e_2). We consider a control volume $\Omega \subset \mathbb{R}^2$ containing the solid for every time $t \geq 0$. Hence, the analysis of the coupled problem is set in $\Omega \times (0, T)$ where $T > 0$ is an arbitrary time.
In order to deal with such coupled system, we introduce a diffeomorphic map sending a fix reference domain Ω_0 into the physical configuration Ω at time $t \geq 0$.
Without loss of generality, we choose the reference configuration to be the physical configuration at initial time $\Omega(t = 0)$.
Hence, we define a map $T_t \in \mathcal{C}^1(\overline{\Omega_0})$ such that

$$\begin{aligned} \overline{\Omega_t^f} &= T_t(\overline{\Omega_0^f}), \\ \overline{\Omega_t^s} &= T_t(\overline{\Omega_0^s}) \end{aligned}$$

Since we only consider one degree of freedom motion, we write

$$\overline{\Omega_t^s} = \overline{\Omega_0^s} + d(t)e_2$$

We set $\Sigma^s \equiv \bigcup\limits_{0<t<T} (\{t\} \times \Gamma_t^s)$, $Q^f \equiv \bigcup\limits_{0<t<T} (\{t\} \times \Omega_t^f)$ and $\Sigma_\infty^f \equiv \Gamma_\infty^f \times (0, T)$
The map T_t can be actually defined as the flow of a particular vector field, as described in the following lemma :

Lemma 1 ([27]) *Assuming that $d(.)$ is smooth enough, there exists a vector field V that builds Q , i.e*

$$\begin{aligned} \overline{\Omega_t^f} &= T_t(V)(\overline{\Omega_0^f}), \\ \overline{\Omega_t^s} &= T_t(V)(\overline{\Omega_0^s}) \end{aligned}$$

where $T_t(V)$ is solution of the following dynamical system :

$$\begin{array}{rrcl} T_t(V): & \Omega_0 & \longrightarrow & \Omega \\ & x_0 & \longmapsto & x(t,x_0) \equiv T_t(V)(x_0) \end{array}$$

with

$$\begin{array}{ll} \dfrac{\mathrm{d}\,x}{\mathrm{d}\,\tau} = V(\tau, x(\tau)), & \tau \in [0,T] \\ x(\tau=0) = x_0, & \mathit{in}\ \Omega_0 \end{array} \tag{1}$$

In our simple case, we can give an example of an appropriate flow vector field :

$$\left\{ \begin{array}{ll} V(x,t) = \dot{d}(t)e_2, & x \in \overline{\Omega^s_t} \\ V(x,t) = \mathrm{Ext}(\dot{d}(t)e_2), & x \in \Omega^f_t \\ V(x,t)\cdot n = 0, & x \in \Gamma^f_\infty \end{array} \right. \tag{2}$$

where Ext is an arbitrary extension operator from Γ^s_0 into Ω^f_0. The map T_t is usually referred as the Arbitrary Euler-Lagrange map.
The solid is described by the evolution of its displacement and its velocity and the couple $(d, \dot{d})$ is solution of the following ordinary second order differential equation :

$$\left\{ \begin{array}{l} m\,\ddot{d} + k\,d = F_f, \\ \left[d, \dot{d}\right](t=0) = [d_0, d_1] \end{array} \right. \tag{3}$$

where (m, k) stand for the structural mass and stiffness. F_f is the projection of the fluid loads on Γ^s_t along the direction of motion e_2.
The fluid is assumed to be a viscous incompressible newtonian fluid. Its evolution is described by its velocity u and its pressure p. The couple (u, p) satisfies the classical Navier-Stokes equations written in non-conservative form :

$$\left\{ \begin{array}{ll} \partial_t u + \mathrm{D}\,u\cdot u - \nu\Delta u + \nabla p = 0, & Q^f(V) \\ \mathrm{div}(u) = 0, & Q^f(V) \\ u = u_\infty, & \Sigma^f_\infty \\ u(t=0) = u_0, & \Omega^f_0 \end{array} \right. \tag{4}$$

where ν stands for the kinematic viscosity and u_∞ is the farfield velocity field.
Hence, the projected fluid loads F_f have the following expression :

$$F_f = -\left(\int_{\Gamma^s_t} \sigma(u,p)\cdot n\right)\cdot e_2 \tag{5}$$

where $\sigma(u,p) = -p\,\mathrm{I} + \nu(\mathrm{D}\,u + {}^*\mathrm{D}\,u)$ stands for the fluid stress tensor inside Ω^f_t, with $(\mathrm{D}\,u)_{ij} = \partial_j u_i = u_{i,j}$.

We complete the whole system with kinematic continuity conditions at the fluid-structure interface Γ_t^s :

$$u = V = \dot{d}\, e_2, \text{ on } \Sigma^s(V) \tag{6}$$

To summarize, we get the following coupled system :

$$\left\{\begin{array}{ll} \partial_t u + \mathrm{D}\, u \cdot u - \nu \Delta u + \nabla p = 0, & Q^f(V) \\ \mathrm{div}(u) = 0, & Q^f(V) \\ u = u_\infty, & \Sigma_\infty^f \\ u = \dot{d}\, e_2, & \Sigma^s(V) \\ m\, \ddot{d} + k\, d = -\left(\displaystyle\int_{\Gamma_t^s} \sigma(u,p) \cdot n \right) \cdot e_2, & (0,T) \\ \left[u, d, \dot{d}\right](t=0) = [u_0, d_0, d_1]\,, & \Omega_0^f \times \mathbb{R}^2 \end{array}\right. \tag{7}$$

Main Result: For $u_\infty \in \mathcal{U}_c$ the following minimization problem,

$$\min_{u_\infty \in \mathcal{U}_c} j(u_\infty) \tag{8}$$

where $j(u_\infty) = J_{u_\infty}(\left[u, p, d, \dot{d}\right](u_\infty))$ with $\left[u, p, d, \dot{d}\right](u_\infty)$ is a weak solution of problem (7) and J_{u_∞} is a real functional of the following form :

$$J_{u_\infty}(\left[u, p, d, \dot{d}\right]) = \frac{\alpha}{2} \int_0^T \left[|d - d_g^1|^2 + |\dot{d} - d_g^2|^2 \right] + \frac{\gamma}{2} \|u_\infty\|_{\mathcal{U}_c}^2 \tag{9}$$

admits at least a solution u_∞^* which satisfies the following necessary first-order optimality conditions,

$$\nabla j(u_\infty^*) = (\sigma(\varphi_{u_\infty^*}, \pi_{u_\infty^*}) \cdot n)|_{\Sigma_\infty^f} + \gamma\, u_\infty^* = 0 \tag{10}$$

with (φ, π, b_1, b_2) are solutions of the following adjoint system,

$$\left\{\begin{array}{ll} -\partial_t \varphi - \mathrm{D}\, \varphi \cdot u + ({}^*Du) \cdot \varphi - \nu \Delta \varphi + \nabla q = 0, & Q^f(V) \\ \mathrm{div}(\varphi) = 0, & Q^f(V) \\ \varphi = b_2\, e_2, & \Sigma^s(V) \\ \varphi = 0, & \Sigma_\infty^f \\ \varphi(T) = 0, & \Omega_T^f \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{ll} -\dot{b}_1 + k\, b_2 = \alpha(d - d_g^1), & (0,T) \\ b_1(T) = 0, & \end{array}\right. \tag{12}$$

$$\left\{\begin{array}{ll} -\partial_t \lambda - \nabla_\Gamma \lambda \cdot V = f, & \Sigma^s(\dot{d}\, e_2) \\ \lambda(T) = 0, & \Gamma_T(\dot{d}\, e_2) \end{array}\right. \tag{13}$$

with $f = \left[-\dot{d}\,\dot{b_2} + \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n) - |\dot{d}|^2\,(\mathrm{D}\,\varphi\cdot e_2)\cdot e_2\right]$ and

$$\int_{\Gamma_t^s(V)} \lambda\, n = \left[-b_1 - m\,\dot{b_2} - \alpha(\dot{d} - d_g^2)\right]\, e_2 + \int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n \tag{14}$$

Remark 1 *We can eliminate the auxiliary adjoint variables* (λ, b_1), *in order to get a system only involving the adjoint variables* (φ,π,b_2) . *We then replace equations* (2), (13) , (14) *by the following second order ODE,*

$$\left\{\begin{array}{ll} m\,\ddot{b}_2 + k\,b_2 = \alpha(d - d_g^1) - \alpha(\dot{d} - \dot{d}_g^2) + \partial_t\left(\displaystyle\int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n\right)\cdot e_2 & \\ + \displaystyle\int_{\Gamma_t^s(V)} \left[|\dot{d}|^2\,(\mathrm{D}\,\varphi\cdot e_2)\cdot e_2 - \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n)\right]\cdot n & (0,T) \\ \left[b_2, \dot{b}_2\right](T) = [0,0]\,, & \end{array}\right. \tag{15}$$

3 Well-posedness of the coupled system

We are interested in recalling global existence results for weak solutions to the initial boundary value problem (7). One should expect some restrictions on the existence time for the solutions since depending on the data, the solid may vanish outside the control volume Ω. This problem was recently investigated by several authors [23], [5], [11], [15], [19], [24]. We refer to [18] and [19] for a complete review. In our case, we only have one degree of freedom for the solid motion and we are dealing with two dimensional Navier-Stokes equations. Nevertheless, we deal with non-homogeneous Dirichlet boundary conditions at the farfield fluid boundary.

Theorem 1 *Assume the following hypothesis :*

i) Ω_0^s, Ω_0 *are of class* $\mathcal{C}^2$,

ii) $a = dist(\Gamma_\infty^f, \Omega_0^s) > 0$,

iii)

$$\begin{array}{lll} u_0 \in (L^2(\Omega_0^f))^2 & u_\infty \in\ (H^m(\Sigma_\infty^f))^2 & m > \frac{3}{4} \\ \mathrm{div}(u_0) = 0, & \textit{in } \Omega_0^f & \\ u_0 = d_1 e_2, & \textit{on } \Gamma_0^s & \end{array} \tag{16}$$

then there exists a positive real time $T_0 = T_0(u_0, a, \Omega_0^s, \Omega_0)$ *such that for any* $T \in\ (0, T_0)$, *there exists at least one weak solution to the initial-boundary value problem (7), with*

$$\begin{array}{c} d \in\ W^{1,\infty}(0,T;\ \mathbb{R}) \cap \mathcal{C}^0([0,T];\ \mathbb{R}) \\ (u,\dot{d}) \in\ L^2(0,T;\ \mathcal{V}_{d(.)}) \cap L^\infty(0,T;\ \mathcal{H}_{d(.)}) \end{array}$$

with

$$\mathcal{H}_{d(t)} \equiv \Big\{(v,\ell) \in (L^2(\Omega_0))^2 \times \mathbb{R}, \quad \operatorname{div}(v) = 0, \\ v \cdot n = 0, \quad \textit{on} \quad \Gamma^f_\infty, \quad v|_{\Omega^s_t} = \ell \cdot e_2,\ supp(v) \subset \Omega^f_t \Big\}$$

and

$$\mathcal{V}_{d(t)} \equiv \Big\{(v,\ell) \in (H^1(\Omega_0))^2 \times \mathbb{R}, \quad \operatorname{div}(v) = 0, \\ v = 0, \quad \textit{on} \quad \Gamma^f_\infty, \quad v|_{\Omega^s_t} = \ell \cdot e_2,\ supp(v) \subset \Omega^f_t \Big\}$$

and satisfies the following identity :

$$\begin{aligned} & - \int_0^T \left[\int_{\Omega^f_t} u \cdot \partial_t v + m\dot{d}\,\dot{\ell} - k\,d\,\ell \right] \\ & + \int_0^T \int_{\Omega^f_t} [(\mathrm{D}\,u \cdot u) \cdot v + \nu\,\mathrm{D}\,u \cdot\cdot \mathrm{D}\,v] = m d_1\,\ell(0) + \int_{\Omega^f_0} u_0 \cdot v(0) \\ & \forall\ (v,\ell) \in\ \mathcal{C}^1([0,T];\ \mathcal{V}_{d(.)})\ \textit{with}\ v(T) = \dot{\ell}(T) = 0 \end{aligned} \tag{17}$$

Remark 2 *Using results from Fursikov et al [17], we can relax the regularity assumption for u_∞ and ask :*

iii)

$$\begin{aligned} u_\infty \cdot \tau \in\ & L^2(0,T;\ (H^{\frac{1}{2}}(\Gamma^f_\infty))^2) \cap H^{\frac{1}{4}}(0,T;\ (L^2(\Gamma^f_\infty))^2) \\ u_\infty \cdot n \in\ & L^2(0,T;\ (H^{\frac{1}{2}}(\Gamma^f_\infty))^2) \cap H^{\frac{3}{4}}(0,T;\ (H^{-1}(\Gamma^f_\infty))^2) \end{aligned}$$

4 Inverse problem settings

As mentionned previously, we would like to perform a mathematical analysis of the Instability Tracking method [21]. More precisely, we analyse the minimization problem involved in this method. In the sequel, we will highly use the framework introduced by Zolésio and its collaborators concerning shape optimization tools [9], [25], [12], [13], [14].
We are interested in solving the following minimization problem :

$$\min_{u_\infty \in \mathcal{U}_c} j(u_\infty) \tag{18}$$

where $j(u_\infty) = J_{u_\infty}(\left[u,p,d,\dot{d}\right](u_\infty))$ with $\left[u,p,d,\dot{d}\right](u_\infty)$ is a weak solution of problem (7) and J_{u_∞} is a real functional of the following form :

$$J_{u_\infty}(\left[u,p,d,\dot{d}\right]) = \frac{\alpha}{2}\int_0^T \left[|d - d^1_g|^2 + |\dot{d} - d^2_g|^2\right] + \frac{\gamma}{2}\|u_\infty\|^2_{\mathcal{U}_c} \tag{19}$$

The main difficulty in dealing with such a minimization problem is related to the dependence of integrals on the unknown domain Ω_t^f which depends also on the control variable u_∞. This point will be solved by using the ALE map T_t introduced previously.

4.1 Analysis strategy

We shall focus our efforts on the derivation of first-order optimality conditions for problem (18). This involves the computation of the gradient with respect to the inflow condition u_∞ of the cost function J_{u_∞}.
There exists several methods to handle such a question,

- Following [13], it is possible to handle the derivative $\frac{D}{Du_\infty}(u,p,d)\cdot\delta u_\infty$ using a back transport map into a fix domain and use the weak implicit function theorem to justify and obtain the linearized system. Once the full linear tangent system is defined, it is possible to define an adjoint system which solution may be involved in the computation of the objective function gradient.

- An other possible choice is to try to pass through the obtention of a linear tangent system and directly get the adjoint system. This may be realized using a Min-Max formulation of the minimization problem (18) taking into account the coupled system constraint through Lagrange multipliers.

In this report, we shall use the latter choice coupled with the introduction a transverse map in order to handle the sensitivity analysis of the Lagrangian functional with respect to variation of the fluid domain.

4.2 Free divergence and non-homogeneous Dirichlet boundary condition constraints

We shall now describe the way to take into account inside the Lagrangian functional, several constraints associated to the coupled system.
The divergence free condition coming from the fact that the fluid has an homogeneous density and evolves as an incompressible flow is difficult to impose on the mathematical and numerical point of view. We choose to include the divergence free condition directly into the Lagrangian functional thanks to a multiplier that may play the role of the adjoint variable associated to the primal pressure variable. This leads in a certain sense to a saddle point formulation or mixed formulation of the Navier-Stokes subsystem. We shall use the following identity,

$$-\int_{\Omega_t^f} q\,\mathrm{div}\,u = \int_{\Omega_t^f} u\cdot\nabla q - \int_{\Gamma_\infty^f} u\cdot(q\,n) - \int_{\Gamma_t^s} u\cdot(q\,n) \tag{20}$$

The coupled system (7) involves two essential non-homogeneous Dirichlet boundary conditions,

$$u = u_\infty, \quad \Sigma_\infty^f \tag{21}$$
$$u = \dot{d}\, e_2, \quad \Sigma_0^s \tag{22}$$

We use a very weak formulation of the state equation, consisting in totally transposing the laplacian operator,

$$\int_{\Omega_t^f} -\nu \Delta u \cdot \phi = \int_{\Omega_t^f} -\nu \Delta \phi \cdot u + \int_{\Gamma_\infty^f \cup \Gamma_t^s} \nu \left[u \cdot \partial_n \phi - \phi \cdot \partial_n u \right] d\Gamma \tag{23}$$

We shall substitute inside this identity the desirable boundary conditions and we will recover the boundary constraints in performing an integration by parts inside the optimality conditions corresponding to the sensitivity with respect to the multipliers. This procedure has been already use in [25] and [8] to perform shape optimization problems for elliptic equations using Min-Max principles.
We shall also choose to transpose the time operator inside the weak formulation. This has to be performed very carefully, since we are dealing with a moving domain,

$$\int_0^T \int_{\Omega_t^f} \partial_t u \cdot v \quad = \quad -\int_0^T \int_{\Omega_t^f} u \cdot \partial_t v - \int_0^T \int_{\partial \Omega_t^f} (u \cdot v) \, \langle V, n \rangle \tag{24}$$
$$+ \int_{\Omega_T^f} u(T) \cdot v(T) - \int_{\Omega_0^f} u(0) \cdot v(0) \tag{25}$$

Remark 3 *This kind of techniques has been popularized in [20] as a systematic way to study non-homogeneous linear partial differential equations. These formulations are usually called very weak formulations or transposition formulations. We shall notice that these methods are still valid in the non-linear case to obtain regularity or existence results. We refer to [2] for a recent applications on the Navier-Stokes system.*

4.3 Solid reduced order and solid weak state operator

For the sake of simplicity, we reduce the order of the solid governing equation by defining the global solid variable,

$$\tilde{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d \\ \dot{d} \end{pmatrix} \tag{26}$$

leading to the first order ordinary differential equation,

$$M \dot{\tilde{d}} + K \tilde{d} = F \tag{27}$$

with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & -1 \\ k & 0 \end{bmatrix}$$

and

$$F = \begin{pmatrix} 0 \\ F_f \end{pmatrix}$$

In the case of the coupled fluid-solid system, the loads depend on the fluid state variable,

$$F_f(u,p) = -\left(\int_{\Gamma_t^s} \sigma(u,p)\cdot n\right)\cdot e_2$$

We introduce the state and multiplier spaces in order to define a proper solid weak state operator,

$$X_1^s = \left\{d_1 \in \mathcal{C}^1([0,T])\right\} \tag{28}$$

$$X_2^s = \left\{d_2 \in \mathcal{C}^1([0,T])\right\} \tag{29}$$

$$Y_1^s = \left\{b_1 \in \mathcal{C}^1([0,T])\right\} \tag{30}$$

$$Y_2^s = \left\{b_2 \in \mathcal{C}^1([0,T])\right\} \tag{31}$$

and the load space,

$$L = \left\{F \in \mathcal{C}^1([0,T])\right\}$$

This allows us to define a solid state operator,

$$e^s : X_1^s \times X_2^s \times L \longrightarrow (Y_1^s \times Y_2^s)^*$$

whose action is defined by the following identity,

$$\begin{aligned}\langle e^s(d_1,d_2,F),(b_1,b_2)\rangle = &\int_0^T \left[-d_1\,\dot{b_1} - d_2\,b_1\right] - d_1^0 b_1(0) + d_1(T)b_1(T)\\ &+ \int_0^T \left[-m\,d_2\,\dot{b_2} + k\,d_1\,b_2\right] - m\,d_2^0\,b_2(0) + m\,d_2(T)\,b_2(T) - \int_0^T F\,b_2\end{aligned}$$

4.4 Fluid state operator

In this section, we summarize the different options that we have chosen for the Lagrangian framework and define the variational state operator constraint.
In order to deal with rigid displacement vector fields, we introduce the following spaces :
Rigid displacement spaces:

$$\begin{aligned}
\mathcal{H}^1 &\stackrel{\text{def}}{=} \{\phi \in (H^1(\Omega_0))^2; \quad \nabla\phi = 0, \quad \text{in} \quad \Omega_0^s\} \\
\mathcal{H}_0^1 &\stackrel{\text{def}}{=} \{\phi \in (H^1(\Omega_0))^2; \quad \nabla\phi = 0, \quad \text{in} \quad \Omega_0^s, \quad \phi = 0, \quad \text{on} \quad \Gamma_\infty^f\}
\end{aligned}$$

Lemma 2 *For $\varphi \in \mathcal{H}^1$, assuming that Ω_0^s is connected, there exists $C \in \mathbb{R}^2$ such that*

$$\varphi|_{\Omega_0^s} = C$$

In the sequel, we will need to define precise state and multiplier spaces in order to endow our problem with a Lagrangian functional framework.
Following the existence result stated previously, we introduce the fluid state space :

$$X^f \stackrel{\text{def}}{=} \left\{u \in H^2(0,T;(H^2(\Omega_t^f))^2 \cap \mathcal{H}^1)\right\}$$

$$Z^f \stackrel{\text{def}}{=} \{p \in H^1(0,T;(H^1(D))^2)\}$$

We also need test function spaces that will be useful to define Lagrange multipliers :

$$\begin{aligned}
Y^f &\stackrel{\text{def}}{=} \left\{v \in L^2(0,T;\,(H^2(\Omega_t^f))^2 \cap \mathcal{H}_0^1)\right\} \\
V^f &\stackrel{\text{def}}{=} \{q \in H^1(0,T;(H^1(D))^2)\} \\
W_s^f &\stackrel{\text{def}}{=} \{(v,b_2) \in Y^f \times Y_2^s,\, v|_{\Gamma_t^s} = b_2 \cdot e_2\}
\end{aligned}$$

We define the fluid weak state operator,

$$e_{u_\infty}^f : X^f \times Z^f \times U^s \longrightarrow (Y^f \times V^f)^*$$

whose action is defined by :

$$\begin{aligned}
\langle e_{u_\infty}^f(u,p,u^s),(v,q)\rangle &= \int_0^T \int_{\Omega_t^f} [-u\cdot\partial_t v + (\mathrm{D}\,u\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla\, q - p\,\mathrm{div}\, v] \\
&+ \int_0^T \int_{\Gamma_\infty^f} u_\infty\cdot(\sigma(v,q)\cdot n) + \int_0^T \int_{\Gamma_t^s} [u^s\cdot(\sigma(v,q)\cdot n) - (u\cdot v)\,\langle u^s, n\rangle] \\
&- \int_0^T \int_{\Gamma_t^s} v\cdot(\sigma(u,p)\cdot n) + \int_{\Omega_T} u(T)\cdot v(T) - \int_{\Omega_0} u_0\cdot v(t=0) \\
&\qquad \forall\,(v,q) \in\; Y^f \times V^f
\end{aligned}$$

4.5 Coupled system operator

Our mechanical system consist of a solid part and a fluid part. These subsystems have been represented thanks to a solid and a fluid state operator. It is now possible to couple these two operators in order to build an ad-hoc coupled system operator.
The major point here, is to notice that the fluid load F_f appears in the fluid state operator and then can be coupled with the solid part thanks to the input load F of the solid operator. To achieve this coupling, we need to decide whether or not the fluid and the solid multipliers match at the fluid-solid interface. If not, we have to work with the fluid constraint tensor at the fluid-solid boundary, what may be not convenient due to regularity requirement. Hence, we choose to work with continuous test functions on Γ_t^s. This means that we shall choose the fluid and second solid multiplier spaces to be the space W_s^f. We define the coupled system weak state operator as follows,

$$e_{u_\infty} : Y^f \times Z^f \times X_1^s \times X_2^s \longrightarrow (W_s^f \times V^f \times Y_1^s)^*$$

whose action is defined by the following identity,

$$\begin{aligned}
\langle e_{u_\infty}(u,p,d_1,d_2),(v,q,b_1,b_2)\rangle &= \langle e^s(d_1,d_2,F_f),(b_1,b_2)\rangle + \langle e^f_{u_\infty}(u,p,d_2\,e_2),(v,q,d_1\cdot e_2)\rangle \\
&= \int_0^T \int_{\Omega_t^f} [-u\cdot\partial_t v + (\mathrm{D}\,u\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla\, q - p\ \mathrm{div}\, v] \\
&\quad + \int_0^T \int_{\Gamma_\infty^f} u_\infty\cdot(\sigma(v,q)\cdot n) + \int_0^T \int_{\Gamma_t^s} [(d_2\,e_2)\cdot(\sigma(v,q)\cdot n) - (u\cdot v)\,\langle d_2\,e_2,n\rangle] \\
&\quad + \int_0^T \left[-d_1\,\dot{b_1} - d_2\,b_1\right] + \int_0^T \left[-m\,d_2\,\dot{b_2} + k\,d_1\,b_2\right] \\
&\quad + \int_{\Omega_T} u(T)\cdot v(T) - \int_{\Omega_0} u_0\cdot v(t=0) - d_1^0\,b_1(0) + d_1(T)\,b_1(T) - md_2^0\,b_2(0) + m\,d_2(T)\,b_2(T) \\
&\qquad \forall\,(v,b_2,q,b_1)\in\ W_s^f\times V^f\times Y_1^s
\end{aligned}$$

4.6 Min-Max problem

In this section, we introduce the lagrangian functional associated with problem (7) and problem (18) :

$$\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2) \stackrel{\mathrm{def}}{=} J_{u_\infty}(u,p,d_1,d_2) - \langle e_{u_\infty}(u,p,d_1,d_2),(v,q,b_1,b_2)\rangle \tag{32}$$

with

$$J_{u_\infty}(u,p,d_1,d_2) = \frac{\alpha}{2}\left[\int_0^T |d_1-d_g^1|^2 + |d_2-d_g^2|^2\right] + \frac{\gamma}{2}\|u_\infty\|^2_{\mathcal{U}_c}$$

Using this functional, the optimal control problem (18) can be put in the following form :

$$\min_{u_\infty \in \mathcal{U}_c} \quad \min_{(u,p,d_1,d_2)\in X^f\times Z^f\times X_1^s\times X_2^s} \quad \max_{(v,b_2,q,b_1)\in W_s^f\times V^f\times Y_1^s} \quad \mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2) \tag{33}$$

By using the Min-Max framework, we avoid the computation of the state derivative with respect to u_∞. First-order optimality conditions will furnish the gradient of the original cost function using the solution of an adjoint problem.
We would like to apply min-max differentiability results to problem (33). The main issue is to prove that the min-max subproblem

$$\min_{(u,p,d_1,d_2)\in X^f\times Z^f\times X_1^s\times X_2^s} \quad \max_{(v,b_2,q,b_1)\in W_s^f\times V^f\times Y_1^s} \quad \mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2) \tag{34}$$

admits at least one saddle point for $u_\infty \in \mathcal{U}_c$.

4.6.1 Reduced Gradient

We assume that the conditions to apply the Min-Max principle [6] are fullfilled so we can bypass the derivation with respect to the control variable u_∞ trough the min-max subproblem (34). It leads to the following result :

Theorem 2 *For $u_\infty \in \mathcal{U}_c$, and $(u_{u_\infty}, p_{u_\infty}, \tilde{d}_{u_\infty}, \varphi_{u_\infty}, \pi_{u_\infty}, \tilde{b}_{u_\infty})$ the unique saddle point of problem (34), the gradient of the cost function j at point $u_\infty \in \mathcal{U}_c$ is given by the following expression :*

$$\nabla j(u_\infty) = (\sigma(\varphi_{u_\infty}, \pi_{u_\infty})\cdot n)|_{\Sigma_\infty^f} + \gamma\, u_\infty \tag{35}$$

Proof : Using theorem (3) from [8], we bypass the derivation with respect to u_∞ inside the min-max subproblem (34) :

$$
\begin{aligned}
&< j^{'}(u_\infty), \delta u_\infty >= \langle \frac{D}{Du_\infty}\mathcal{L}_{u_\infty}(u_{u_\infty}, p_{u_\infty}, d^1_{u_\infty}, d^2_{u_\infty}; \phi_{u_\infty}, \pi_{u_\infty}, b^1_{u_\infty}, b^2_{u_\infty}), \delta u_\infty\rangle \\
&= \min_{(u,p,d_1,d_2)\in X^f\times Z^f\times X_1^s\times X_2^s} \quad \max_{(v,b_2,q,b_1)\in W_s^f\times V^f\times Y_1^s} \langle \frac{D}{Du_\infty}\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2), \delta u_\infty\rangle \\
&\qquad = \int_0^T \int_{\Gamma_\infty^f} (\nu\, \partial_n \phi_{u_\infty} - \pi_{u_\infty}\, n)\cdot \delta u_\infty + \gamma \int_0^T \int_{\Gamma_\infty^f} u_\infty \cdot \delta u_\infty
\end{aligned}
$$

□

5 KKT Optimality Conditions

In this section, we are interested in establishing the first order optimality condition for problem (34), better known as Karusch-Kuhn-Tucker optimality conditions. This step is crucial, because it leads to the formulation of the adjoint problem satisfied by the Lagrange multipliers $(\varphi_{u_\infty}, \pi_{u_\infty}, b^1_{u_\infty}, b^2_{u_\infty})$. We recall the expression of the Lagrangian,

$$\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2) \stackrel{\text{def}}{=} J_{u_\infty}(u,p,d_1,d_2) - \langle e_{u_\infty}(u,p,d_1,d_2),(v,q,b_1,b_2)\rangle \tag{36}$$

The KKT system will have the following structure :

$$\begin{array}{rcl} \partial_{(v,q,b_1,b_2)}\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2)\cdot(\delta v,\delta q,\delta b_1,\delta b_2) & = & 0, \\ \forall\,(\delta v,\delta b_2,\delta q,\delta b_1)\in\ W^f_s\times V^f\times Y^s_1 \rightarrow & & \text{State Equations} \\ \partial_{(u,p,d_1,d_2)}\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2)\cdot(\delta u,\delta p,\delta d_1,\delta d_2) & = & 0, \\ \forall\,(\delta u,\delta p,\delta d_1,\delta d_2)\in X^f\times Z^f\times X^s_1\times X^s_2 \rightarrow & & \text{Adjoint Equations} \end{array}$$

5.1 Derivatives with respect to state variables

5.1.1 Fluid adjoint system

Lemma 3 *For $u_\infty\in\mathcal{U}_c$, $(p,b_1,b_2,v,b_2,q,b_1)\in Z^f\times X^s_1\times X^s_2\times W^f_s\times V^f\times Y^s_1$, $\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2)$ is differentiable with respect to $u\in Y^f$ and we have*

$$\begin{aligned}\langle\partial_u\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2),\,\delta u\rangle = \\ -\int_0^T\int_{\Omega^f_t}[-\delta u\cdot\partial_t v+[\mathrm{D}\,\delta u\cdot u+\mathrm{D}\,u\cdot\delta u]\cdot v-\nu\delta u\cdot\Delta v+\delta u\cdot\nabla q] \\ +\int_0^T\int_{\Gamma^s_t}(\delta u\cdot v)\,\langle d_2\,e_2,n\rangle-\int_{\Omega_T}\delta u(T)\cdot v(T)\quad\forall\delta u\in\ X^f\end{aligned}$$

In order to obtain a strong formulation of the fluid adjoint problem, we perform some integration by parts :

Lemma 4

$$\int_{\Omega^f_t}(\mathrm{D}\,\delta u\cdot u)\cdot v \;=\; -\int_{\Omega^f_t}[\mathrm{D}\,v\cdot u+\mathrm{div}(u)\,v]\cdot\delta u+\int_{\Gamma^f_\infty\cup\Gamma^s_t}(\delta u\cdot v)\,\langle u,n\rangle$$

It leads to the following identity :

$$\begin{aligned}\langle\partial_{\hat u}\mathcal{L}_{u_\infty}(u,p,d_1,d_2;\varphi,\pi,b_1,b_2),\,\delta u\rangle = \\ -\int_{Q^f}[-\partial_t\varphi+({}^*Du)\cdot\varphi-(D\varphi)\cdot u-\mathrm{div}(u)\,\varphi-\nu\Delta\varphi+\nabla\pi]\cdot\delta u-\int_{\Omega^f_T}\varphi(T)\cdot\delta u(T)\end{aligned}$$

Lemma 5 *For $u_\infty\in\mathcal{U}_c$, $(u,b_1,b_2,v,b_2,q,b_1)\in X^f\times X^s_1\times X^s_2\times W^f_s\times V^f\times Y^s_1$, $\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2)$ is differentiable with respect to $p\in Z^f$ and we have*

$$\langle\partial_p\mathcal{L}_{u_\infty}(\hat u,p,d_1,d_2;\varphi,q,b_1,b_2),\,\delta p\rangle=\int_0^T\int_{\Omega^f_t}\delta p\;\mathrm{div}\,\varphi,\quad\forall\delta p\in\ Z^f$$

This leads to the following fluid adjoint strong formulation,

$$\left\{\begin{array}{ll} -\partial_t \varphi - \mathrm{D}\,\varphi \cdot u + ({}^*Du)\cdot\varphi - \nu\Delta\varphi + \nabla q = 0, & Q^f \\ \mathrm{div}(\varphi) = 0, & Q^f \\ \varphi = b_2 \cdot e_2, & \Sigma^s \\ \varphi = 0, & \Sigma^f_\infty \\ \varphi(T) = 0, & \Omega^f_T \end{array}\right. \tag{37}$$

5.2 Solid adjoint system

We recall that the ALE map is built as the flow a vector field V that matches the solid velocity at the fluid-solid interface and is arbitrary inside the fluid domain, i.e using the reduced oder model,

$$\left\{\begin{array}{ll} V(x,t) = d_2 \cdot e_2, & x \in \overline{\Omega^s_t} \\ V(x,t) = \mathrm{Ext}(d_2 \cdot e_2), & x \in \Omega^f_t \\ V(x,t)\cdot n = 0, & x \in \Gamma^f_\infty \end{array}\right. \tag{38}$$

Hence, the ALE map depends on d_2 through V. Furthermore, the derivative with respect to d_1 may be simpler since it does not involve derivative with respect to the geometry. Then, we have the following result,

Lemma 6 *For $u_\infty \in \mathcal{U}_c$ and $(u,p,d_1,d_2,v,b_2,q,b_1) \in X^f \times Z^f \times X^s_1 \times X^s_2 \times W^f_s \times V^f \times Y^s_1$, the Lagrangian $\mathcal{L}_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2)$ is differentiable with respect to $d_1 \in X^s_1$ and we have*

$$\langle \partial_{d_1}\mathcal{L}_{u_\infty}(u,p,d_1,d_2;\varphi,\pi,b_1,b_2),\, \delta d_1\rangle = \int_0^T \left[\alpha\,(d_1 - d^1_g)\,\delta d_1 - k\,\delta d_1\, b_2 + \delta d_1\, \dot{b}_1\right] - \delta d_1(T)\, b_1(T)$$

From which we deduce the following adjoint ODE,

$$\left\{\begin{array}{ll} -\dot{b_1} + k\, b_2 = \alpha(d_1 - d^1_g), & (0,T) \\ b_1(T) = 0, & \end{array}\right. \tag{39}$$

The derivative of the Lagrangian with respect to the solid velocity variable d_2 involves shape derivatives of domain integrals.
This point has been investigated previously by Zolésio in [29], [28] and in [14], [12]. Then we need to introduce the concept of Transverse Field associated to a perturbation of the solid velocity.
We introduce a perturbation flow W associated to the perturbation $\delta d_2 e_2$. For example,

$$W = \mathrm{Ext}(\delta d_2 \cdot e_2)$$

This flow generates new fluid and solid domains through the ALE map, $T_t(V+\rho W)$, with $\rho \geq 0$. We set

$$
\begin{aligned}
\Omega_t^{f,\rho} &\stackrel{\text{def}}{=} T_t(V+\rho W)(\Omega_0^f) \\
\Omega_t^{s,\rho} &\stackrel{\text{def}}{=} T_t(V+\rho W)(\Omega_0^s)
\end{aligned}
$$

For the sake of simplicity, we set

$$T_t^\rho \stackrel{\text{def}}{=} T_t(V+\rho W)$$

The objective of this paragraph is to compute the following derivative :

$$\left(\frac{d}{d\rho}\mathcal{L}_{u_\infty}(\hat{u}, p, d_1, d_2+\rho\delta d_2; \hat{v}, q, b_1, b_2)\right)\Big|_{\rho=0}$$

5.2.1 Transverse map and vector field

Since we would like to differentiate the Lagrangian functional with respect to ρ at point $\rho = 0$, it is convenient to work with function already defined in $\Omega_t^{f,\rho=0} \stackrel{\text{def}}{=} \Omega_t^f$ as we proceed to the limit $\rho \to 0$.
To this end, we introduce a new map as in [29] :

$$
\mathcal{T}_\rho^t \stackrel{\text{def}}{=} T_t^\rho \circ T_t^{-1} : \begin{array}{ccc} \Omega_t^f & \longrightarrow & \Omega_t^{f,\rho} \\ \Omega_t^s & \longrightarrow & \Omega_t^{s,\rho} \end{array} \tag{40}
$$

This map is actually the flow of a vector field following for $\rho \in (0, \rho_0)$,

Theorem 3 *The Transverse map T_t^ρ is the flow of a transverse field $\mathcal{Z}_\rho^t$ defined as follow :*

$$\mathcal{Z}_\rho^t \stackrel{\text{def}}{=} \mathcal{Z}^t(\rho, .) = \left(\frac{\partial \mathcal{T}_\rho^t}{\partial \rho}\right) \circ (\mathcal{T}_\rho^t)^{-1} \tag{41}$$

i.e is the solution of the following dynamical system :

$$
\begin{array}{rcl} T_t^\rho(\mathcal{Z}_\rho^t) : \quad \Omega & \longrightarrow & \Omega \\ x & \longmapsto & x(\rho, x) \equiv T_t^\rho(\mathcal{Z}_\rho^t)(x) \end{array}
$$

with

$$
\begin{aligned}
&\frac{\mathrm{d}\, x(\rho)}{\mathrm{d}\, \rho} = (\rho, x(\rho)), \quad && \rho \geq 0 \\
&x(\rho = 0) = x, && \mathit{in}\ \Omega
\end{aligned} \tag{42}
$$

Since, we only consider derivatives at point $\rho = 0$, we set $Z_t \stackrel{\text{def}}{=} \mathcal{Z}^t_{\rho=0}$. We recall a result from [13] which might be useful for the sequel,

Theorem 4 *The mapping,*

$$\begin{array}{lll} [0,\rho_0] & \longrightarrow & \mathcal{C}^0([0,T]; W^{k-1,\infty}(\Omega)) \\ \rho & \longmapsto & T_t(V+\rho W) \end{array}$$

is continuously differentiable and

$$S^t(\rho,.) \stackrel{\text{def}}{=} \partial_\rho \left[T_t(V+\rho W)\right] = \int_0^t [\mathrm{D}(V+\rho W)\circ T_\tau(V+\rho W)\cdot S^\tau(\rho,.) \\ +W\circ T_\tau(V+\rho W) d\tau] \tag{43}$$

Corollary 1 $S_t(.) \stackrel{\text{def}}{=} S^t(\rho=0,.)$ *is the unique solution of the following Cauchy problem,*

$$\left\{ \begin{array}{ll} \partial_t S_t - (\mathrm{D}\, V\circ T_t)\cdot S_t = W\circ T_t, & \Omega_0\times(0,T) \\ S_{t=0} = 0, & \Omega_0 \end{array} \right. \tag{44}$$

A fundamental result furnish the dynamical system satisfied by the vector field Z_t related to the vector fields (V, W),

Theorem 5 ([14]) *The vector field* Z_t *is the unique solution of the following Cauchy problem,*

$$\left\{ \begin{array}{ll} \partial_t Z_t + [Z_t, V] = W, & \Omega_0\times(0,T) \\ Z_{t=0} = 0, & \Omega_0 \end{array} \right. \tag{45}$$

where $[Z_t, V] \stackrel{\text{def}}{=} DZ_t\cdot V - DV\cdot Z_t$ *stands for the Lie bracket of the pair* (Z_t, V).

5.2.2 Shape derivatives

In the sequel, we will need general results concerning shape derivatives of integral over domains or boundaries. We will use the framework developed in Sokolowski-Zolésio [25].

Lemma 7

$$\frac{d}{d\rho}\left(\int_{\Omega_t^{f,\rho}} G(\rho)\, d\Omega\right)\Bigg|_{\rho=0} = \int_{\Omega_t^f} \partial_\rho G(\rho)\, d\Omega + \int_{\Gamma_t^s} G(\rho=0)\langle Z_t, n\rangle\, d\Gamma \tag{46}$$

Lemma 8

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^{s,\rho}} \phi(\rho)\, d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t^s} \left[\phi'_\Gamma + H\phi\,\langle Z_t, n\rangle\,\right]\, d\Gamma \tag{47}$$

where ϕ'_Γ *stands for the tangential shape derivative of* $\phi(\rho,.)\in L^1(\Gamma_t^s)$

We recall classical definitions of shape derivative functions :

Definition 1 *For $\phi(\rho,x) \in \mathcal{C}^0((0,\rho_0;\mathcal{C}^1(\Gamma_t^{s,\rho}))$, the material derivative is given by the following expression :*

$$\dot{\phi} = \frac{d}{d\rho}\left(\phi(\rho,.)\circ\mathcal{T}_\rho^t\right)\Big|_{\rho=0}$$

then the tangential shape derivative of ϕ is given by the following expression,

$$\phi_\Gamma' \stackrel{\text{def}}{=} \dot{\phi} - \nabla_\Gamma\,\phi\cdot Z_t$$

Remark 4 *If ϕ is the trace of a vector field $\tilde{\phi}$ defined over Ω, then we have,*

$$\phi_\Gamma' \stackrel{\text{def}}{=} \tilde{\phi}'|_{\Gamma_t^s} + \partial_n\tilde{\phi}\,\langle Z_t,n\rangle$$

with $\tilde{\phi}'|_{\Gamma_t^s} \stackrel{\text{def}}{=} \dfrac{d}{d\,\rho}\left(\tilde{\phi}(\rho,.)\right)\Big|_{\rho=0}\Big|_{\Gamma_t^s}$.

following this remark, we have,

Lemma 9

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^{s,\rho}}\tilde{\phi}(\rho,x)da\right)\Bigg|_{\rho=0} = \int_{\Gamma_t^s}\left[\tilde{\phi}' + \left[H\tilde{\phi}+\partial_n\tilde{\phi}\right]\,\langle Z_t,n\rangle\right]\,d\Gamma \tag{48}$$

5.2.3 Derivation of the perturbed Lagrangian

Thanks to the introduction of the transverse map, it is now possible to work with functions (u,v) that are defined on Ω_t^f. This leads to the following perturbed Lagrangian :

$$\begin{aligned}
&\mathcal{L}_{u_\infty}^\rho \stackrel{\text{def}}{=} \mathcal{L}_{u_\infty}(u,p,d_1,d_2+\rho\delta d_2;v,q,b_1,b_2)\\
&= J_{u_\infty}(u,p,d_1,d_2+\rho\delta d_2) - \Bigg[\int_0^T\int_{\Omega_t^{f,\rho}}\big[-(u\circ\mathcal{R}_\rho^t)\cdot\partial_t(v\circ\mathcal{R}_\rho^t) + (\mathrm{D}\,u\circ\mathcal{R}_\rho^t\cdot u\circ\mathcal{R}_\rho^t)\cdot v\circ\mathcal{R}_\rho^t\\
&\qquad -\nu u\circ\mathcal{R}_\rho^t\cdot\Delta v\circ\mathcal{R}_\rho^t + u\circ\mathcal{R}_\rho^t\cdot\nabla\,q - p\,\operatorname{div} v\circ\mathcal{R}_\rho^t\big] + \int_0^T\int_{\Gamma_\infty^f} u_\infty\cdot(\sigma(v,q)\cdot n)\\
&\quad +\int_0^T\int_{\Gamma_t^{s,\rho}}\big[((d_2+\rho\delta d_2)\,e_2)\cdot(\sigma(v\circ\mathcal{R}_\rho^t,q)\cdot n^\rho) - u\circ\mathcal{R}_\rho^t\cdot v\circ\mathcal{R}_\rho^t\cdot((d_2+\rho\delta d_2)\,e_2)\cdot n^\rho\big]\\
&\qquad +\int_0^T\left[-d_1\,\dot{b_1} - (d_2+\rho\delta d_2)\,b_1\right] + \int_0^T\left[-m\,(d_2+\rho\delta d_2)\,\dot{b_2} + k\,d_1\,b_2\right]\\
&\qquad +\int_{\Omega_T}u(T)\cdot v(T) - \int_{\Omega_0}u_0\cdot v(t=0) - d_1^0\,b_1(0) + d_1(T)\,b_1(T) - m\,d_2^0\,b_2(0)\\
&\qquad\qquad +m\,(d_2+\rho\delta d_2)(T)\,b_2(T)\big]\quad\forall\,(v,b_2,q,b_1)\in\,W_s^f\times V^f\times Y_1^s
\end{aligned}$$

with $\mathcal{R}_\rho^t \stackrel{\text{def}}{=} (\mathcal{T}_\rho^t)^{-1}$. We apply the previous results to the perturbed Lagrangian functional $\mathcal{L}_{u_\infty}$.

a) Distributed terms:
We set,

$$G(\rho,.) = \big[-u\circ\mathcal{R}_\rho^t \cdot \partial_t(v\circ\mathcal{R}_\rho^t) + D(u\circ\mathcal{R}_\rho^t)\cdot(u\circ\mathcal{R}_\rho^t)\cdot v\circ\mathcal{R}_\rho^t \\ -\nu(u\circ\mathcal{R}_\rho^t)\cdot\Delta(v\circ\mathcal{R}_\rho^t) + (u\circ\mathcal{R}_\rho^t)\cdot\nabla q - p\ \mathrm{div}(v\circ\mathcal{R}_\rho^t)\big]$$

with $\mathcal{R}_\rho^t \stackrel{\text{def}}{=} (\mathcal{T}_\rho^t)^{-1}$.
We need the following lemmas in order to derivate $G(\rho,.)$ with respect to ρ,

Lemma 10

$$\left(\frac{d\mathcal{T}_\rho^t}{d\rho}\right)\Big|_{\rho=0} = Z_t$$

$$\left(\frac{d\mathcal{R}_\rho^t}{d\rho}\right)\Big|_{\rho=0} = -Z_t$$

Lemma 11

$$\left(\frac{d\left(u\circ\mathcal{R}_\rho^t\right)}{d\rho}\right)\Big|_{\rho=0} = -\,\mathrm{D}\,u\cdot Z_t$$

Proof :
Using the chain rule we get

$$\begin{aligned}\left(\frac{d}{d\rho}\left(u\circ\mathcal{R}_\rho^t\right)\right)\Big|_{\rho=0} &= \left(\mathrm{D}\,u\circ\mathcal{R}_\rho^t\right)\cdot\left(\frac{\mathrm{D}\,\mathcal{R}_\rho^t}{\mathrm{D}\,\rho}\right)\Big|_{\rho=0} \\ &= -\left(\mathrm{D}\,u\circ\mathcal{R}_\rho^t\right)\cdot\mathcal{Z}^t(\rho,.)\big|_{\rho=0} \\ &= -\,\mathrm{D}\,u\cdot Z_t\end{aligned}$$

□

Lemma 12 *Then, we have the following result*

$$\partial_\rho G(\rho,.)|_{\rho=0} = [(\mathrm{D}\,u\cdot Z_t)\cdot\partial_t v + u\cdot(\partial_t(\mathrm{D}\,v\cdot Z_t)) \\ -[(\mathrm{D}(\mathrm{D}\,u\cdot Z_t))\cdot u + \mathrm{D}\,u\cdot(\mathrm{D}\,u\cdot Z_t)]\cdot v - (\mathrm{D}\,u\cdot u)\cdot(\mathrm{D}\,v\cdot Z_t) \\ +\nu(\mathrm{D}\,u\cdot Z_t)\cdot\Delta v + \nu u\cdot(\Delta(\mathrm{D}\,v\cdot Z_t)) + p\ \mathrm{div}(\mathrm{D}\,v\cdot Z_t) - (\mathrm{D}\,u\cdot Z_t)\cdot\nabla q]$$

Proof :
It comes easily using definition of $G(\rho,.)$ and lemma (10)-(11).
□
Then we have an expression of the derivative of distributed terms coming from the Lagrangian with respect to ρ,

$$\frac{d}{d\rho}\left(\int_{\Omega_t^{f,\rho}} G(\rho)d\Omega\right)\Bigg|_{\rho=0} = \int_{\Omega_t^f} [(\mathrm{D}\,u\cdot Z_t)\cdot\partial_t v + u\cdot(\partial_t(\mathrm{D}\,v\cdot Z_t))$$
$$-[(\mathrm{D}(\mathrm{D}\,u\cdot Z_t))\cdot u + \mathrm{D}\,u\cdot(\mathrm{D}\,u\cdot Z_t)]\cdot v - (\mathrm{D}\,u\cdot u)\cdot(\mathrm{D}\,v\cdot Z_t)$$
$$+\nu(\mathrm{D}\,u\cdot Z_t)\cdot\Delta v + \nu u\cdot(\Delta(\mathrm{D}\,v\cdot Z_t)) + p\ \mathrm{div}(\mathrm{D}\,v\cdot Z_t) - (\mathrm{D}\,u\cdot Z_t)\cdot\nabla q]$$
$$+\int_\Gamma [-u\cdot\partial_t v + (Du\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla q - p\ \mathrm{div}(v)]\,\langle Z_t, n\rangle$$

b) Boundary terms :
We must now take into account the terms coming from the moving boundary $\Gamma_t^{s,\rho}$. Then we set,

$$\phi(\rho,.) \;=\; \big[(d_2+\rho\delta d_2)\,e_2)\cdot(\sigma(v\circ\mathcal{R}_\rho^t,q) - u\circ\mathcal{R}_\rho^t\cdot v\circ\mathcal{R}_\rho^t\cdot((d_2+\rho\delta d_2)\,e_2)\big]\;\cdot n^\rho$$

We set $V = d_2\,e_2$ and $W = \delta d_2\,e_2$, then

$$\phi(\rho,.) = (V+\rho W)\cdot\big[\sigma(v\circ\mathcal{R}_\rho^t,q) - u\circ\mathcal{R}_\rho^t\cdot v\circ\mathcal{R}_\rho^t\big]\cdot n^\rho$$

Since $\phi(\rho,.)$ is defined on the boundary $\Gamma_t^{s,\rho}$, we need some extra identities corresponding to boundary shape derivates of terms involved in $\phi(\rho,.)$.

Lemma 13 ([10])

$$\partial_\rho n^\rho|_{\rho=0} = n_\Gamma' = -\nabla_\Gamma(Z_t\cdot n)$$

Lemma 14

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^{s,\rho}}\langle E(\rho), n^\rho\rangle d\Gamma\right)\Bigg|_{\rho=0} \;=\; \int_{\Gamma_t^s}\langle E'|_\Gamma, n\rangle + (\mathrm{div}\,E)\langle Z_t, n\rangle$$

Proof :
First, we use that,

$$\int_{\Gamma_t^{s,\rho}}\langle E(\rho), n^\rho\rangle = \int_{\Omega_t^{f,\rho}}\mathrm{div}\,E(\rho)$$

then we derive this quantity using lemma (7),

$$\frac{d}{d\rho}\left(\int_{\Omega_t^{f,\rho}} \operatorname{div} E(\rho)\right)\Bigg|_{\rho=0} = \int_{\Omega_t^f} \operatorname{div} E' + \int_{\Gamma_t^s} (\operatorname{div} E)\langle Z_t, n\rangle$$

We conclude using $\displaystyle\int_{\Omega_t^f} \operatorname{div} E' = \int_{\Gamma_t^s} \langle E', n\rangle$.

□

Then we use,

Lemma 15

$$E'|_\Gamma = W \cdot [-q\,\mathrm{I} + \nu\,\mathrm{D}\,v - u \cdot v] + V \cdot [-\nu\,\mathrm{D}(\mathrm{D}\,v \cdot Z_t) + (\mathrm{D}\,u \cdot Z_t) \cdot v + u \cdot (\mathrm{D}\,v \cdot Z_t)] \tag{49}$$

Hence, we have

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^{s,\rho}} \phi(\rho)\,d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t^s(V)} W \cdot [-q\,\mathrm{I} + \nu\,\mathrm{D}\,v - u \cdot v] \cdot n$$

$$+ \int_{\Gamma_t^s(V)} V \cdot [-\nu\,\mathrm{D}(\mathrm{D}\,v \cdot Z_t) + (\mathrm{D}\,u \cdot Z_t) \cdot v + u \cdot (\mathrm{D}\,v \cdot Z_t)] \cdot n$$

$$+ \int_{\Gamma_t^s(V)} \operatorname{div}(V \cdot [-q\,\mathrm{I} + \nu^*\,\mathrm{D}\,v - u \cdot v])\langle Z_t, n\rangle$$

Remark 5 *We recall that,*

$$\begin{aligned}
\int_\Gamma V \cdot (\mathrm{D}\,v \cdot n) &= \int_{\Omega_t^f} \operatorname{div}({}^*\,\mathrm{D}\,v \cdot V) \\
&= \int_{\Omega_t^f} \mathrm{D}\,v \cdot\cdot\, \mathrm{D}\,V + V \cdot \Delta v
\end{aligned} \tag{50}$$

We use this expression in the sequel. We recall that the perturbed lagrangian has the following form,

$$\begin{aligned}
\mathcal{L}_{V,W}^\rho = J_{V,W}^\rho - \int_0^T \int_{\Omega_t^{f,\rho}} G(\rho) - \int_0^T \int_{\Gamma_t^{s,\rho}} \phi(\rho) \\
- \int_{\Omega_T} u(T) \cdot v(T) + \int_{\Omega_0} u_0 \cdot v(t=0) \\
\forall\,(v,q) \in\ Y(\Omega_t^f) \times V^f
\end{aligned}$$

Hence, its derivative with respect to ρ at point $\rho = 0$ has the following expression,

$$\frac{d}{d\rho}\left(\mathcal{L}^{\rho}_{V,W}\right)\Big|_{\rho=0} = \frac{d}{d\rho}\left(J^{\rho}_{V,W}\right)\Big|_{\rho=0} - \int_0^T \frac{d}{d\rho}\left(\int_{\Omega_t^{f,\rho}} G(\rho)\right)\Big|_{\rho=0} - \int_0^T \frac{d}{d\rho}\left(\int_{\Gamma_t^{s,\rho}} \phi(\rho)\right)\Big|_{\rho=0}$$
$$\forall\,(v,q) \in\ Y(\Omega_t^f) \times V^f$$

Furthermore we have,

Lemma 16

$$\frac{d}{d\rho}\left(J^{\rho}_{V,W}\right)\Big|_{\rho=0} = \int_0^T \alpha(d_2 - d_g^2)\, e_2 \cdot W \tag{51}$$

Using the last identities concerning the derivative of the distributed and the boundary terms with respect to ρ, we get the following expression,

$$\frac{d}{d\rho}\left(\mathcal{L}^{\rho}_{V,W}\right)\Big|_{\rho=0} = -A_{Z_t} - B_{Z_t} - C_W \tag{52}$$

with

$$A_{Z_t} = \int_0^T \int_{\Omega_t^f(V)} [(\mathrm{D}\, u \cdot Z_t) \cdot \partial_t v - [(\mathrm{D}(\mathrm{D}\, u \cdot Z_t)) \cdot u$$
$$+ \mathrm{D}\, u \cdot (\mathrm{D}\, u \cdot Z_t)] \cdot v + \nu(\mathrm{D}\, u \cdot Z_t) \cdot \Delta v - (\mathrm{D}\, u \cdot Z_t) \cdot \nabla q]$$
$$+ \int_0^T \int_{\Omega_t^f(V)} [u \cdot (\partial_t(\mathrm{D}\, v \cdot Z_t)) - (\mathrm{D}\, u \cdot u) \cdot (\mathrm{D}\, v \cdot Z_t) + \nu u \cdot (\Delta(\mathrm{D}\, v \cdot Z_t)) + p\ \mathrm{div}(\mathrm{D}\, v \cdot Z_t)]$$

$$B_{Z_t} = \int_0^T \int_{\Gamma_t^s(V)} [-u \cdot \partial_t v + (Du \cdot u) \cdot v - \nu u \cdot \Delta v + u \cdot \nabla q - p\ \mathrm{div}(v)]\,(Z_t \cdot n)$$
$$+ V \cdot [-\nu\, \mathrm{D}(\mathrm{D}\, v \cdot Z_t) + (\mathrm{D}\, u \cdot Z_t) \cdot v + u \cdot (\mathrm{D}\, v \cdot Z_t)] \cdot n$$
$$+ \mathrm{div}(V \cdot [-q\, \mathrm{I} + \nu^*\, \mathrm{D}\, v - u \cdot v])\langle Z_t, n\rangle$$
$$- \int_0^T \int_{\Gamma_\infty^f} \nu\, u_\infty \cdot \mathrm{D}(\mathrm{D}\, v \cdot Z_t) \cdot n$$

$$C_W = \int_0^T \int_{\Gamma_t^s(V)} W \cdot \left[-b_1\, e_2 - m\, \dot{b_2}\, e_2 + \sigma(v,q) \cdot n - (u \cdot v)\, n - \alpha(d_2 - d_g^2)\, e_2\right]$$

5.2.4 The shape derivative kernel identity

We shall now, assume that (u, p, φ, π) is a saddle point of the Lagrangian functional $\mathcal{L}_{u_\infty}$. This will help us to simplify several terms involved in the derivative of $\mathcal{L}_{u_\infty}$ with respect to V.
Indeed, we would like to express the distributed term A_{Z_t} as a boundary quantity defined on the fluid moving boundary Γ_t^s and the fixed boundary Γ_∞^f.

Theorem 6 *For (u, p, φ, π) saddle points of the Lagrangian functional (32), the following identity holds,*

$$\int_0^T \int_{\Omega_t^f(V)} [(\mathrm{D}\, u \cdot Z_t) \cdot \partial_t \varphi - [(\mathrm{D}(\mathrm{D}\, u \cdot Z_t)) \cdot u$$

$$+ \mathrm{D}\, u \cdot (\mathrm{D}\, u \cdot Z_t)] \cdot \varphi + \nu(\mathrm{D}\, u \cdot Z_t) \cdot \Delta\varphi - (\mathrm{D}\, u \cdot Z_t) \cdot \nabla\pi]$$

$$+ \int_0^T \int_{\Omega_t^f(V)} [u \cdot (\partial_t(\mathrm{D}\, \varphi \cdot Z_t)) - (\mathrm{D}\, u \cdot u) \cdot (\mathrm{D}\, \varphi \cdot Z_t) + \nu u \cdot (\Delta(\mathrm{D}\, \varphi \cdot Z_t)) + p\ \mathrm{div}(\mathrm{D}\, \varphi \cdot Z_t)]$$

$$- \int_0^T \int_{\Gamma_t^s(V)} V \cdot [\nu\, \mathrm{D}(\mathrm{D}\, \varphi \cdot Z_t) - (\mathrm{D}\, u \cdot Z_t) \cdot \varphi - u \cdot (\mathrm{D}\, \varphi \cdot Z_t)] \cdot n$$

$$+ \int_0^T \int_{\Gamma_t^s(V)} [\nu\, (\varphi - b_2\, e_2) \cdot (\mathrm{D}(\mathrm{D}\, u \cdot Z_t) \cdot n) + (\mathrm{D}\, \varphi \cdot Z_t) \cdot (-p\, n + \nu(\mathrm{D}\, u \cdot n))]$$

$$- \int_0^T \int_{\Gamma_\infty^f} \nu\, u_\infty \cdot (\mathrm{D}(\mathrm{D}\, \varphi \cdot Z_t) \cdot n) = 0, \quad \forall\, W \stackrel{\text{def}}{=} \delta d_2\, e_2$$

Proof :
We use extremal conditions associated to variations with respect to (u, v) in the Lagrangian functional where we have added a boundary since we consider test functions v that do not vanish on Γ_∞^f and do not match the solid test functions at the fluid-solid interface $\Gamma_t^s(V)$,

i.e

$$\mathcal{L}^2_{u_\infty}(u,p,d_1,d_2;v,q,b_1,b_2) = J_{u_\infty}(u,p,d_1,d_2)$$
$$-\int_0^T\int_{\Omega_t^f(V)}[-u\cdot\partial_t v + (\mathrm{D}\,u\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla\, q - p\ \mathrm{div}\, v]$$
$$-\int_0^T\int_{\Gamma^f_\infty} u_\infty\cdot\sigma(v,q)\cdot n - \int_0^T\int_{\Gamma^s_t(V)} V\cdot[\sigma(v,q)\cdot n - (u\cdot v)\,n] + \int_0^T\int_{\Gamma^s_t(V)}(v - b_2\,e_2)\cdot(\sigma(u,p)\cdot n)$$
$$-\int_0^T\left[-d_1\,\dot{b_1} - d_2\,b_1\right] - \int_0^T\left[-md_2\,\dot{b_2} + kd_1\,b_2\right]$$
$$-\int_{\Omega_T} u(T)v(T) + \int_{\Omega_0} u_0 v(t=0) + d_1^0 b_1(0) - d_1(T)b_1(T) + md_2^0 b_2(0) - md_2(T)b_2(T)$$
$$\forall\,(v,q,b_1,b_2)\in\ Y^f\times V^f\times Y_1^s\times Y_2^s$$

This leads to the following perturbation identity,

$$\partial_{(u,v)}\mathcal{L}^2_{u_\infty}\cdot(\delta u,\delta v) = -\int_0^T\int_{\Omega_t^f}[-\delta u\cdot\partial_t v - u\cdot\partial_t\delta v + D(\delta u\cdot u)\cdot v + D(u\cdot\delta u)\cdot v$$
$$+D(u\cdot u)\cdot\delta v - \nu(\delta u\cdot\Delta v) - \nu(u\cdot\Delta\delta v) + \delta u\cdot\nabla q - p\ \mathrm{div}(\delta v)] - \int_0^T\int_{\Gamma^f_\infty}\nu\,u_\infty\cdot(\mathrm{D}\,\delta v\cdot n)$$
$$+\int_0^T\int_{\Gamma^s_t(V)}[\nu\,(v - b_2\,e_2)\cdot(\mathrm{D}\,\delta u\cdot n) + \delta v\cdot(-p\,n + \nu(\mathrm{D}\,u\cdot n))]$$
$$-\int_0^T\int_{\Gamma^s_t(V)} V\cdot[\nu\,\mathrm{D}(\delta v) - \delta u\cdot v - u\cdot\delta v]\cdot n - \int_{\Omega_T}[\delta u(T)v(T) + u(T)\delta v(T)]$$
$$\forall\,(\delta u,\delta v)\in\ X(\Omega_t^f)\times Y(\Omega_t^f)$$

We choose specific perturbation directions, i.e

$$\delta u = \mathrm{D}\,u\cdot Z_t \quad \delta v = \mathrm{D}\,v\cdot Z_t$$

with $\delta u(T) = \delta v(T) = \delta u(0) = \delta v(0) = 0$, where (u,v) are saddle points of the lagrangian, i.e solutions of respectively the primal and adjoint fluid problem. We recognize immediately the distributed and boundary terms involved in the shape derivative kernel identity.
□

5.2.5 Solid and ALE adjoint problem

Using the shape derivative kernel identity, we simplify the Lagrangian derivative at the saddle point $(u, p, d_1, d_2, \varphi, b_1, b_2)$. We set $(u, v) = (u, \varphi)$ and we get,

$$\begin{aligned}
A_{Z_t} = &\int_0^T \int_{\Gamma_t^s(V)} V \cdot [\nu\,\mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) - (\mathrm{D}\,u \cdot Z_t)\cdot\varphi - u\cdot(\mathrm{D}\,\varphi\cdot Z_t)]\cdot n \\
&- \int_0^T \int_{\Gamma_t^s(V)} [\nu\,(\varphi - b_2\,e_2)\cdot(\mathrm{D}(\mathrm{D}\,u\cdot Z_t)\cdot n) + (\mathrm{D}\,\varphi\cdot Z_t)\cdot(-p\,n + \nu(\mathrm{D}\,u\cdot n))] \\
&+ \int_0^T \int_{\Gamma_\infty^f} \nu\,u_\infty\cdot(\mathrm{D}(\mathrm{D}\,\varphi\cdot Z_t)\cdot n)
\end{aligned}$$

We use that $\varphi = b_2\,e_2,$ on $\Gamma_t^s(V)$ and the following identities,

$$(\mathrm{D}\,\varphi\cdot Z_t)\cdot(p\,n) = (p\ \mathrm{div}\,\varphi)\langle Z_t, n\rangle \tag{53}$$

$$(\mathrm{D}\,\varphi\cdot Z_t)\cdot(\mathrm{D}\,u\cdot n) = (\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n)\langle Z_t, n\rangle \tag{54}$$

then,

$$\begin{aligned}
A_{Z_t} = &\int_0^T \int_{\Gamma_t^s(V)} V \cdot [\nu\,\mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) - (\mathrm{D}\,u \cdot Z_t)\cdot\varphi - u\cdot(\mathrm{D}\,\varphi\cdot Z_t)]\cdot n \\
&- \int_0^T \int_{\Gamma_t^s(V)} [-p\ \mathrm{div}\,\varphi + \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n)]\,\langle Z_t, n\rangle \\
&+ \int_0^T \int_{\Gamma_\infty^f} \nu\,u_\infty\cdot(\mathrm{D}(\mathrm{D}\,\varphi\cdot Z_t)\cdot n)
\end{aligned}$$

Using the following identity,

$$\begin{aligned}
\mathrm{div}(V\cdot[-\pi\,\mathrm{I} + \nu^*\,\mathrm{D}\,\phi - u\cdot\varphi]) = &-\pi\ \mathrm{div}\,V - V\cdot\nabla\,\pi + \nu\,\mathrm{D}\,\varphi\cdot\cdot\,\mathrm{D}\,V \\
&+ \nu\,V\Delta\varphi - (\mathrm{div}\,V)\,u\cdot\varphi - V\cdot\nabla(u\cdot\varphi)
\end{aligned} \tag{55}$$

we get,

$$B_{Z_t} = \int_0^T \int_{\Gamma_t^s(V)} [-u \cdot \partial_t \varphi + (Du \cdot u) \cdot \varphi - \nu u \cdot \Delta\varphi + u \cdot \nabla\pi - p \ \mathrm{div}(\varphi)] \langle Z_t, n\rangle$$

$$+ V \cdot [-\nu\, \mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) + (\mathrm{D}\, u \cdot Z_t) \cdot \varphi + u \cdot (\mathrm{D}\,\varphi \cdot Z_t)] \cdot n$$

$$\begin{array}{c}[-\pi \ \mathrm{div}\, V - V \cdot \nabla\,\pi + \nu\, \mathrm{D}\,\varphi \cdot\cdot\, \mathrm{D}\, V + \nu\, V \Delta\varphi \\ -(\mathrm{div}\, V)\, u \cdot \varphi - V \cdot \nabla(u \cdot \varphi)] \langle Z_t, n\rangle\end{array}$$

$$- \int_0^T \int_{\Gamma_\infty^f} \nu\, u_\infty \cdot \mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) \cdot n$$

then we get,

$$\begin{aligned} A_{Z_t} + B_{Z_t} = \int_0^T \int_{\Gamma_t^s(V)} & [\nu\, (\mathrm{D}\,\varphi \cdot n) \cdot (\mathrm{D}(V-u) \cdot n) - u \cdot \partial_t \varphi + (Du \cdot u) \cdot \varphi - \pi \ \mathrm{div}\, V \\ & -(\mathrm{div}\, V)\, u \cdot \varphi - V \cdot \nabla(u \cdot \varphi)] \langle Z_t, n\rangle \end{aligned}$$

We use the following identity,

$$V \cdot \nabla(u \cdot \varphi) = \varphi \cdot \mathrm{D}\, u \cdot V + V \cdot \mathrm{D}\,\varphi \cdot u$$

and the boundary conditions, $u = d_2\, e_2$ on Γ_t^s, $\varphi = b_2\, e_2$ on Γ_t^s. This leads to,

$$\begin{aligned} A_{Z_t} + B_{Z_t} = \int_0^T \int_{\Gamma_t^s(V)} & \Big[\nu\, (\mathrm{D}\,\varphi \cdot n) \cdot (\mathrm{D}(V-u) \cdot n) - d_2\, \dot{b_2} - \pi \ \mathrm{div}\, V - (\mathrm{div}\, V)\, u \cdot \varphi \\ & -d_2\, e_2 \cdot \mathrm{D}\,\varphi \cdot d_2\, e_2] \langle Z_t, n\rangle \end{aligned}$$

We choose the velocity field $V = \mathrm{Ext}(d_2\, e_2) \circ p$, then

$$(\mathrm{D}\, V \cdot n) \cdot n|_{\Gamma_t^s} = 0$$

and

$$\begin{array}{rcl} \mathrm{div}\, V|_{\Gamma_t^s} & = & \mathrm{div}_\Gamma\, V + (\mathrm{D}\, V \cdot n) \cdot n \\ & = & 0 \end{array}$$

Finally, we have

$$A_{Z_t} \ + \ B_{Z_t} \ = \ \int_0^T \int_{\Gamma_t^s(V)} \Big[-d_2\, \dot{b_2} + \nu\, (\mathrm{D}\,\varphi \cdot n) \cdot (\mathrm{D}\, u \cdot n) - |d_2|^2\, (\mathrm{D}\,\varphi \cdot e_2) \cdot e_2\Big] \langle Z_t, n\rangle$$

and

$$C_W = \int_0^T \int_{\Gamma_t^s(V)} W \cdot \left[-b_1\, e_2 - m\, \dot{b_2}\, e_2 + \sigma(\phi, \pi) \cdot n - \alpha(d_2 - d_g^2)\, e_2 \right] \tag{56}$$

where we have used that,

$$\int_{\Gamma_t^s(V)} d_2\, e_2 \cdot b_2\, e_2 \cdot n = 0$$

We introduce the adjoint field λ solution of the following system,

$$\left\{ \begin{array}{ll} -\partial_t \lambda - \nabla_\Gamma\, \lambda \cdot V - \lambda\, \mathrm{div}_\Gamma\, V = f, & (0,T) \\ \lambda(T) = 0, & \Gamma_T(V) \end{array} \right. \tag{57}$$

with

$$f = \left[-d_2\, \dot{b_2} + \nu\, (\mathrm{D}\, \varphi \cdot n) \cdot (\mathrm{D}\, u \cdot n) - |d_2|^2\, (\mathrm{D}\, \varphi \cdot e_2) \cdot e_2 \right] \tag{58}$$

Remark 6 *In our case* $\mathrm{div}_\Gamma\, V = 0$

We recall the following property,

Lemma 17 ([14],[12],[21]) *For any $E(V) \in L^2(\Sigma^s(V))$ and $(V, W) \in \mathcal{U}_{ad}$, the following identity holds,*

$$\int_0^T \int_{\Gamma_t^s(V)} E(V)\, \langle Z_t, n \rangle = - \int_0^T \int_{\Gamma_t^s(V)} \lambda \langle W, n \rangle \tag{59}$$

where $\lambda \in \mathcal{C}^0([0,T]; H^1(\Gamma_t^s))$ is the unique solution of problem (57) with $f = E$.

Then we have,

$$\begin{aligned} A_{Z_t} + B_{Z_t} &= \int_0^T \int_{\Gamma_t^s(V)} f \langle Z_t, n \rangle \\ &= - \int_0^T \int_{\Gamma_t^s(V)} \langle \lambda\, n, W \rangle \end{aligned}$$

However, using the optimality condition for the Lagrangian functional, we obtain

$$-(A_{Z_t} + B_{Z_t}) = C_W$$

this leads to,

$$\int_0^T \int_{\Gamma_t^s(V)} \lambda\, n \cdot W = \int_0^T \int_{\Gamma_t^s(V)} \left[-b_1\, e_2 - m\, \dot{b_2}\, e_2 + \sigma(\phi, \pi) \cdot n - \alpha(d_2 - d_g^2)\, e_2 \right] \cdot W, \quad \forall\, W \stackrel{\mathrm{def}}{=} \delta d_2\, e_2 \tag{60}$$

From which we deduce the following identity,

$$\int_{\Gamma_t^s(V)} \lambda\, n = \left[-b_1 - m\,\dot{b_2} - \alpha(d_2 - d_g^2)\right]\, e_2 + \int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n \tag{61}$$

We now use the following lemma,

Lemma 18

$$\partial_t \left(\int_{\Gamma_t^s(V)} \lambda\, n\right) = \int_{\Gamma_t^s(V)} \left[\partial_t \lambda + \nabla_\Gamma\, \lambda \cdot V\right]\, n \tag{62}$$

We get,

$$\partial_t \left(\left[-b_1 - m\,\dot{b_2} - \alpha(d_2 - d_g^2)\right]\, e_2 + \int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n\right) = \\ \int_{\Gamma_t^s(V)} \left[-d_2\,\dot{b_2} + \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n) - |d_2|^2\,(\mathrm{D}\,\varphi\cdot e_2)\cdot e_2\right]\, n$$

This can be written as,

$$\dot{b}_1 + m\,\ddot{b}_2 = -\alpha(\dot{d}_2 - \dot{d}_g^2) + \partial_t\left(\int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n\right)\cdot e_2 + \\ \int_{\Gamma_t^s(V)} \left[|d_2|^2\,(\mathrm{D}\,\varphi\cdot e_2)\cdot e_2 - \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n)\right]\cdot n$$

and we recall the other solid adjoint equation,

$$\left\{\begin{array}{l} -\dot{b_1} + k\,b_2 = \alpha(d - d_g^1), \quad (0,T) \\ b_1(T) = 0, \end{array}\right. \tag{63}$$

Let us inject $\dot{b_1}$ inside the first one,

$$m\,\ddot{b}_2 + k\,b_2 = \alpha(d - d_g^1) - \alpha(\dot{d}_2 - \dot{d}_g^2) + \partial_t\left(\int_{\Gamma_t^s(V)} \sigma(\phi,\pi)\cdot n\right)\cdot e_2 + \\ \int_{\Gamma_t^s(V)} \left[|d_2|^2\,(\mathrm{D}\,\varphi\cdot e_2)\cdot e_2 - \nu\,(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,u\cdot n)\right]\cdot n$$

This concludes the proof of the main result.
□

6 Conclusion

In this report, we have investigated sensitivity analysis for a simple 2-D coupled fluid-structure system. This analysis was performed using a Lagrangian functional and non-cylindrical shape derivative tools to handle pertubation with respect to the velocity of the solid. This led to the derivation of first-order optimality conditions for an optimal control problem related to a tracking functionnal. This optimality system can be numerically approximated and included inside a gradient based optimization procedure. This point is under investigation following the strategy adopted for Navier-Stokes optimal control problems as in [16].
The methodology used in this report can be generalized to more complex fluid-structure interaction problems. We can either change the fluid model and handle compressibility as in [4], or change the solid equations and use a real 3D non-linear elastic model [22] or a shell model [3].

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http://www.inria.fr
ISSN 0249-6399